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On the stability of viscous flow between rotating cylinders

Part 2. Numerical analysis

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A simple numerical method is presented for solving the eigenvalue problem which governs the stability of Couette flow. The method is particularly useful in obtaining the eigenfunctions associated with the various modes of instability. When the cylinders rotate in opposite directions, these eigenfunctions exhibit an exponentially damped oscillatory behaviour for sufficiently large values of $-\mu$, where $\mu = \Omega_2/\Omega_1$. In terms of the stream function which describes the motion in planes through the axis of the cylinders, this means that weak, viscously driven cells appear in the outer layers of the fluid which, according to Rayleigh's criterion, are dynamically stable. For $\mu = -3$, for example, four cells are present, the amplitudes of which are in the ratios 1·0: 0·0172: 0·013: 0·00125.

1. Introduction

The problem of the stability of viscous flow between contra-rotating cylinders has been discussed in the preceding paper by an asymptotic method which is especially suited to dealing with the limiting case of $\mu \to -\infty$. For finite negative values of μ, the method is still applicable but the required calculations become excessively laborious. In this paper, therefore, we present an alternative method of dealing with the problem based upon direct numerical integration of the governing equation. This method is particularly well suited to problems of this and similar type in which one wishes to explore the dependence of the solution on one or more parameters.

For purposes of the present discussion it is convenient to write the governing equation in the form (cf. Duty & Reid 1964, equation (18))

$$(D^2-\alpha^2)^3 v = -\alpha^2 \tau (1-z)\, v, \tag{1}$$

where v is the azimuthal component of the perturbation velocity and the radial component of the perturbation velocity u is proportional to $(D^2-\alpha^2)v$. The boundary conditions that must be satisfied are

$$v = (D^2-\alpha^2)v = D(D^2-\alpha^2)v = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad 1-\mu. \tag{2}$$

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In this formulation of the problem the nodal point has been fixed at $z = 1$ and it is only the fluid lying between $z = 0$ and $z = 1$ that is dynamically unstable. As a result of this normalization, α_c and τ_c tend to finite limits as $\mu \to -\infty$.

2. Method of solution

The eigenvalue problem defined by equations (1) and (2) is a two-point boundary-value problem. For numerical purposes, however, it is desirable to convert it into an initial-value problem and this can be done in the following manner. Let us first rewrite equation (1) as a system of first-order equations in the form

$$\left.\begin{aligned} &DU = V, \quad DV = \alpha^2 U + W, \quad DW = X, \\ &DX = \alpha^2 W + Y, \quad DY = Z, \quad DZ = \alpha^2 Y - \alpha^2\tau(1-z)\,U, \end{aligned}\right\} \tag{3}$$

where
$$\left.\begin{aligned} &U = v, \quad V = Dv, \quad W = (D^2-\alpha^2)\,v, \\ &X = D(D^2-\alpha^2)\,v, \quad Y = (D^2-\alpha^2)^2\,v, \quad Z = D(D^2-\alpha^2)^2\,v. \end{aligned}\right\} \tag{4}$$

The boundary conditions (2) then require that

$$U = W = X = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1-\mu. \tag{5}$$

Let us now define three linearly independent solutions V_i $(i = 1, 2, 3)$ of equation (1) by imposing the initial conditions

$$V_i = (D^2-\alpha^2)\,V_i = D(D^2-\alpha^2)\,V_i = 0 \quad (i = 1, 2, 3) \tag{6}$$

and
$$[DV_i,\,(D^2-\alpha^2)^2 V_i,\,D(D^2-\alpha^2)^2 V_i] = \left\{\begin{aligned} &(1,0,0) \quad \text{for} \quad i = 1, \\ &(0,1,0) \quad \text{for} \quad i = 2, \\ &(0,0,1) \quad \text{for} \quad i = 3, \end{aligned}\right\} \tag{7}$$

at $z = 0$.

A solution of equation (1) which automatically satisfies the boundary conditions at $z = 0$ can therefore be written in the form

$$v = \sum_{i=1}^{3} A_i V_i, \tag{8}$$

and the condition that v satisfy the boundary conditions at $z = 1-\mu$ then leads to the relations

$$\sum_{i=1}^{3} A_i V_i = 0, \quad \sum_{i=1}^{3} A_i (D^2-\alpha^2)\,V_i = 0, \quad \sum_{i=1}^{3} A_i D(D^2-\alpha^2)\,V_i = 0. \tag{9}$$

If the constants A_i are not to vanish identically, then it is necessary that the determinant of the system vanish, i.e.

$$\Delta(\alpha, \tau; \mu) \equiv \begin{vmatrix} V_i \\ (D^2-\alpha^2)\,V_i \\ D(D^2-\alpha^2)\,V_i \end{vmatrix} = 0, \tag{10}$$

where the elements in the determinant are evaluated at the outer boundary $z = 1-\mu$. This is the required characteristic equation from which the curves of neutral stability and hence the critical values of α and τ can be obtained.

The basic solutions V_i were obtained by integrating the system of first-order equations (3) by the Runge–Kutta method. The effect of finite interval size was investigated by obtaining solutions at different values of h and then applying Richardson's deferred approach to the limit which is of the order of h^4 in this case. Round-off error was controlled by employing double-precision arithmetic on an IBM 704 computer. It is felt, therefore, that the results given in table 1 (except for those values printed in bold) are correct to within 0·6 unit in the final digit.

μ	α_c	τ_c	a_c	T_c
		First mode		
+1·0	—	—	3·117	1,707·762
0·0	3·1266	3,389·901	3·127	3,389·901
−0·5	**2·133**	**1,266·911**	**3·199**	**6,413·738**
−1·0	1·9995	1,166·412	3·999	18,662·59
−1·5	**2·040**	**1,180·443**	**5·099**	**46,111·06**
−2·0	2·0336	1,178·638	6·101	95,469·68
−2·5	2·0337	1,178·594	7·118	176,862·8
−3·0	2·0337	1,178·596	8·135	301,720·6
		Second mode		
+1·0	—	—	5·365	17,610·39
0·0	**5·361**	**38,497·89**	**5·361**	**38,497·89**
−0·5	**4·134**	**22,693·18**	**6·201**	**114,884·2**
−1·0	4·1824	22,887·95	8·365	366,207·2
−2·0	4·1833	22,894·27	12·55	1854,436

The values shown in bold were obtained by Mr T. H. Hughes on an IBM 7070 computer using single-precision arithmetic; all other values were obtained on an IBM 704 computer using double-precision arithmetic.

TABLE 1. Critical values of the Taylor number and wave-number for the first and second modes of instability.

For a given value of μ, we are primarily interested in determining the least positive value of τ and the corresponding value of α, for these values define the critical conditions at which instability first sets in. This was done by first choosing a value of α and then varying τ until equation (10) was satisfied. This procedure was then repeated for different values of α until the minimum point on the neutral curve was well defined.† With α_c and τ_c determined in this manner, the values of A_2/A_1 and A_3/A_1 could then be determined from any two of the equations (9). Finally, the eigenfunctions thus obtained were normalized so that the amplitude of the radial component of the velocity perturbation was unity. Some results for the second mode of instability have also been obtained. These results not only contribute to our general understanding of the nature of the solutions of equation (1) for negative values of μ but also are relevant in connexion with some of the current ideas on finite-amplitude effects.

† For this problem the neutral curves have only a single minimum. This need not always be the case, however (Hughes & Reid 1962).

3. Results for the first and second modes

The behaviour of α_c and τ_c for negative values of μ (for the first mode of instability) are shown in figures 1 and 2.† The highly damped, oscillatory nature of these curves as $\mu \to -\infty$ is particularly noteworthy. This rather curious behaviour can be traced to the fact that as μ varies from 0 to $-\infty$, the number of

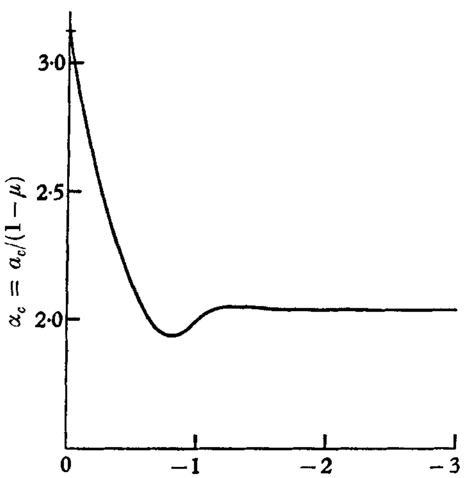

FIGURE 1. The behaviour of the critical wave-number α_c.

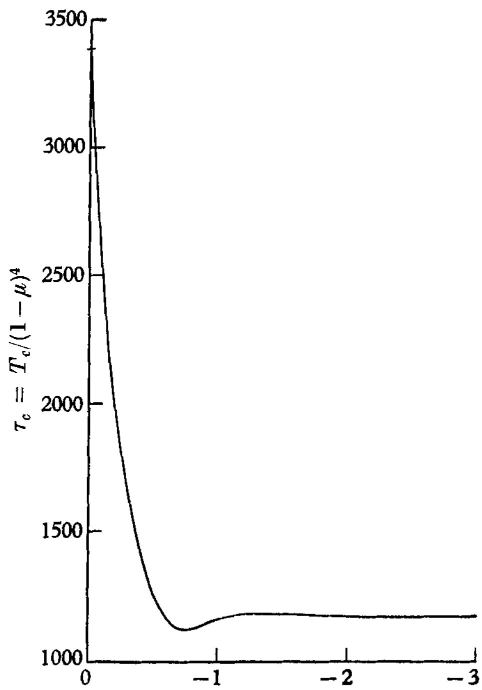

FIGURE 2. The behaviour of the critical Taylor number τ_c.

zeros possessed by the eigenfunctions of the nth mode increase from $n-1$ to ∞. Since the streamfunction which describes the motion in planes through the axis is proportional to the radial eigenfunction, this means that even for the first mode there may be more than one cell across the gap. These outer cells, being located in a region that is dynamically stable, are viscously driven and hence have small amplitudes.

These remarks are illustrated in figures 3–5 which show the radial eigenfunctions for the first‡ and second modes for $\mu = 0$, $-0{\cdot}5$ and $-1{\cdot}0$. In the case of the first mode, this eigenfunction acquires its first zero at a value of μ lying between $-0{\cdot}5$ and $-1{\cdot}0$ and this may partly explain why these viscously driven

† In drawing these figures, the results obtained by Chandrasekhar (1954) have also been used.

‡ The results obtained here for the first mode for $\mu = 0$ are in close agreement with the results obtained recently by Davey (1962). The small differences that do exist between the two sets of results are due entirely to the use of slightly different values of α_c.

outer cells have not been observed experimentally. Unfortunately, even with the present accuracy, it has not been possible in figure 5 to complete the curve for the second eigenfunction. Finally, the cell pattern for $\mu = -3$ is shown in figure 6. This figure clearly shows the heavily damped character of the motion in the outer, stable region of the flow and further illustrates the limitations of Fourier expansion techniques at such large negative values of μ.

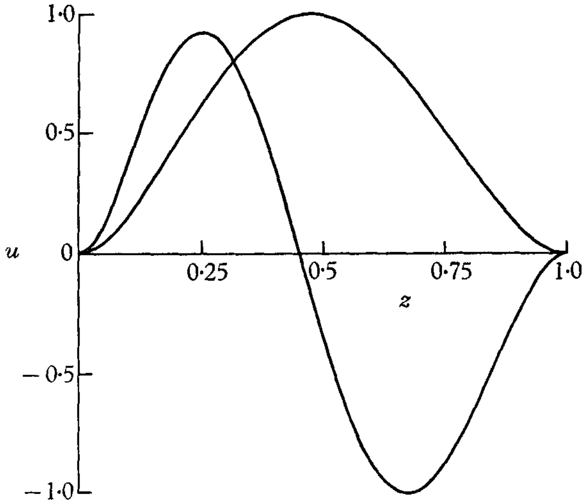

FIGURE 3. The radial eigenfunctions u for the first and second modes of instability for $\mu = 0$.

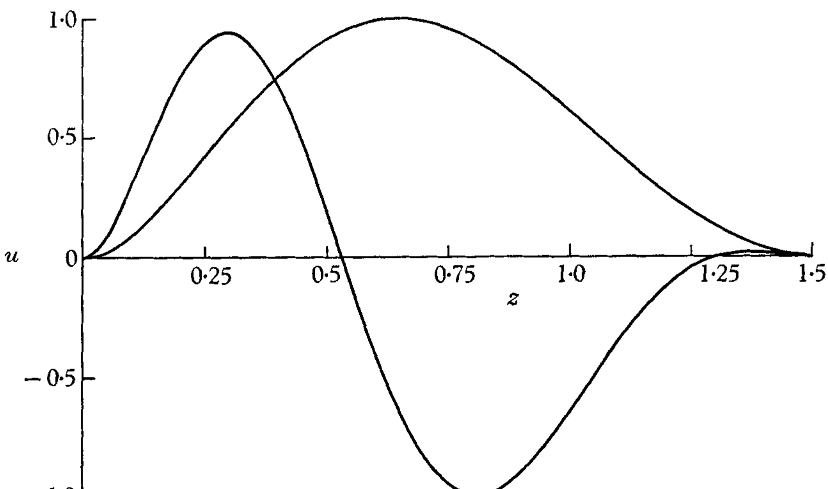

FIGURE 4. The radial eigenfunctions u for the first and second modes of instability for $\mu = -0·5$.

4. Concluding remarks

The method of solution described in this paper is a very simple one, well adapted for use on a high-speed computer. No attempt has been made, however, to achieve fully automatic operation nor have we explored the refinements

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suggested by Fox (1957). Rather we have sought to present the method in its simplest form and to demonstrate its practical usefulness in one particular case. The method does not suffer from some of the limitations of existing methods of approximation; it can, for example, be easily adapted to treat the present problem without making the small-gap approximation.

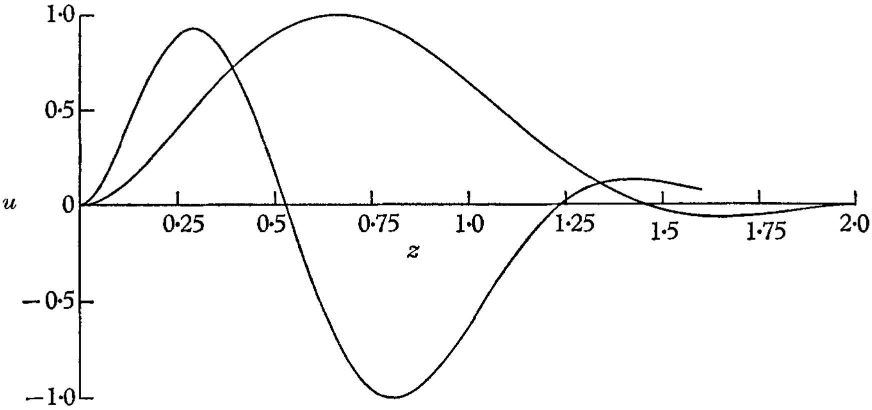

FIGURE 5. The radial eigenfunctions u for the first and second modes of instability for $\mu = -1{\cdot}0$.

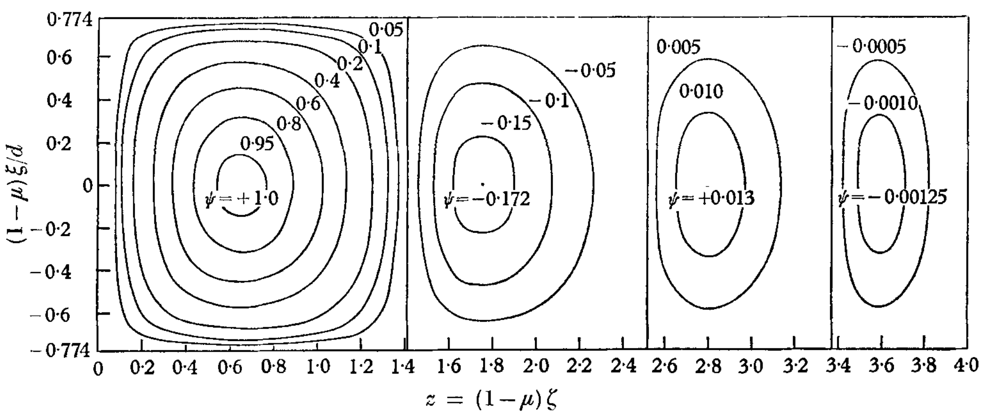

FIGURE 6. The cell pattern at the onset of instability for $\mu = -3$. $\psi = u(z)\cos(a\xi/d)$.

The success of the method lies, in part, in the fact that for flows with curved streamlines there are no critical layers—at least not if one invokes the principle of exchange of stabilities—as there are for parallel flows whose stability is governed by the Orr–Sommerfeld equation. The chief limitation of the method, however, arises from the fact that, even for only moderately negative values of μ, the solutions V_i defined by the initial conditions (6) and (7) tend to become linearly dependent as the outer boundary is approached. More specifically, if eight decimal digits are used in the calculations, then one is limited to a Taylor number T (*not* τ) up to about 10^5 (cf. table 1).

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The untimely death of Daniel Lester Harris, III on 29 April 1962 deprived the second author of a close friend and valued collaborator. The present paper is a brief account of our efforts during the previous three years to develop numerical methods for problems in hydrodynamic stability.

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